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An investigation of the symmetry and singularity properties of classes of third-order fluid problems

S. Jamal* and J.T. Kubayi

Abstract. Several classes of nonlinear differential equations are studied that feature third-order derivatives and have special connections to models in boundary layer theory. We consider the integrability of the equations, which is intimately linked to the singularity structure of its solutions.

In lieu of this, we apply singularity analysis to these models to demonstrate the utility of the method, not only in testing for integrability, but also to achieve a selection method for the free parameters of the models. In particular, we demonstrate how the effects of integrability requirements imposes constraints on the equation.

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1. Introduction

The reduction of partial differential equations to ordinary differential equations is well documented. In particular, many fluid models reduce to third-order equations or instead may be related via some transformation. Some special cases worth mentioning are Prandtl's boundary layer equations that reduce to a third-order equation, namely, the stream function for an incompressible, steady two-dimensional flow with uniform or vanishing mainstream velocity [11]

$$u_y u_{xy} - u_x u_{yy} = v u_{yyy},$$

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and likewise for radial flow, [14]

$$\frac{1}{r}u_z u_{rz} - \frac{1}{r^2}(u_z)^2 - \frac{1}{r}u_r u_{zz} = \nu u_{zzz},\ \nu \text{ constant.}$$

Another example is that of Blasius flow of a steady fluid [6],

$$u_{yyy} + u_x u_{yy} - u_y u_{xy} = 0,$$

and

$$u_{xxx} - 54u^2 u_x - 9u_x^2 - \frac{9}{2}u u_{xx} = u_t,$$

whose reduction is linked to the Chazy IX equation [8].

Many more examples can be given, but instead we present some broad and general forms of such third-order ordinary differential equations that have links to partial differential equations in fluids or boundary flow problems.

Consider the class of equations, where $y = y(x)$ and prime denotes differentiation with respect to the independent variable x:

$$hy''' + byy'' + cy'^2 + ky' = 0, \qquad h, b, c, k \text{ are constants, } h \neq 0, \tag{1}$$

the general class

$$y''' + byy'' + cy'^2 + dy^2 y' + fyy' = 0,\ b, c, d, f \text{ are constants,} \tag{2}$$

the related class of equations

$$y''' + byy'' + cy'^2 - n(ly' - qy^2)^2 = 0,\ b, c, n, l, q \text{ are constants,} \tag{3}$$

and lastly, a third-order class with nonconstant coefficients

$$y''' + b(x)y'' + c(x)y' + d\frac{y'^2}{y} + ky^n = 0,\ d, k, n \text{ are constants.} \tag{4}$$

Aside from fluids, there exist many other cases where the partial differential equation has important connections to an ordinary differential equation, see [12, 23, 24] for example. Similar and related papers on such analysis can be found in [21, 13, 7, 18].

Many of the equations discussed in this work are based on a subsystem of Navier-Stokes system of equations, whereby in boundary layer theory, for instance, similarity variables are derived that lead to a third-order ordinary differential equation. Equations of this nature are commonly analyzed from a numerical perspective.

The precise interpretation of the solution of differential equations may be given in several ways.

One way, is the idea of symmetry induced invariant solutions of a differential equation, and a second approach is singularity analysis. The latter can be traced back to Painlevé [26] after its success in application by Sophie Kowalevskaya [17]. Ablowitz et al. [2, 3, 4] developed an algorithm, called the ARS (Ablowitz-Ramani-Segur) algorithm that tests whether the solution of an ordinary differential equation can be expressed in terms of a Laurent expansion. If so, the equation is said to pass the Painlevé test and is conjectured to be integrable. A detailed description of the algorithm can be found in the work by Conte [10].

The plan of the paper is as follows. In Section 2, all theoretical considerations are discussed. In the sequel, we look at the third-order ordinary boundary flow problems with respect to the singularity algorithm and their symmetries. In Section 4, we summarize the work presented in the paper.

2. Symmetry and singularity theory

The procedure for determining point symmetries for an arbitrary system of equations is as follows [25].

Consider q unknown functions u^α which depend on p independent variables x^i, i.e. $u = (u^1, \dots, u^q)$, $x = (x^1, \dots, x^p)$, with indices $\alpha = 1, \dots, q$ and $i = 1, \dots, p$.

Let

$$G_\alpha\left(x, u^{(k)}\right) = 0, \tag{5}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x.

Definition 2.1. A one-parameter Lie group of transformations (ϵ as the group parameter) that is invariant under (5) is given by

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \tag{6}$$

Invariance of (5) under the transformation (6) implies that any solution $u = \Theta(x)$ of (5) maps into another solution $v = \Psi(x; \epsilon)$ of (5). Expanding (6) around the identity $\epsilon = 0$, generates the following infinitesimal transformations:

$$\begin{aligned}\bar{x}^i &= x^i + \epsilon\xi^i(x, u) + \mathcal{O}(\epsilon^2),\\ \bar{u}^\alpha &= u^\alpha + \epsilon\eta^\alpha(x, u) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{7}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables.

Hence, we consider the vector field $X = \xi^i\partial_{x^i} + \eta^\alpha\partial_{u^\alpha}$.

Definition 2.2. The infinitesimal criterion for invariance is given by

$$X\left[G_\alpha\left(x, u^{(k)}\right)\right] = 0, \quad \text{when} \quad G_\alpha\left(x, u^{(k)}\right) = 0, \tag{8}$$

where X is extended to all derivatives appearing in the equation through an appropriate prolongation.

Regarding singularity analysis, consider the equation

$$y^{(n)} = E\left(x; y'; y'', ..., y^{n-1}\right). \tag{9}$$

If a movable singularity exists, then the solution of (9) will be described by the power- law function $y(x) \simeq (x - x_0)^p$, where p is a negative number and x_0 indicates the singularity's position. It is the initial conditions, that provide us with different positions for the singular point.

The algorithm can be described by the following three steps. The first step is to determine the leading-order behaviour of the dependent variables of the equation.

Hence, we substitute $y(x) = a_0(x - x_0)^p$, where a_0 is a constant and is the coefficient of the leading-order term, into (9) and look for two or more dominant terms. One way to determine the dominant terms is to look for balance after the above substitution by considering the powers in the equation. This provides the value of p followed by a_0.

After obtaining a_0, we look at the "next-to-leading-order" terms, which involves the rest of the leading-order coefficients, the a_κ. In the ARS algorithm we require that the Laurent series be an increasing series called the right Painlevé series

$$y(x) = \sum_{\kappa=0}^{\infty} a_\kappa (x - x_0)^{\kappa+p}, \tag{10}$$

where the singularity at x_0 is a pole of order p.

A decreasing series called the left Painlevé series is of the form

$$y(x) = \sum_{\kappa=0}^{\infty} a_\kappa (x - x_0)^{-\kappa+p}. \tag{11}$$

Next, one must locate the powers at which the arbitrary constants needed to make the solution a general solution, can be introduced. An expression is considered,

$$ny(x) = a_0(x - x_0)^p + m(x - x_0)^{p+s}, \tag{12}$$

where m is a constant, and where the coefficient of terms linear in m is required to be zero. This leads to a determining equation for the exponent s, called the resonance. The solution of $s = -1$ always occurs as it is associated with the movable singularity. If the rest of the values of the resonance s are not integral, then the ARS algorithm terminates.

Lastly, we substitute a truncated Laurent series into the original equation, to check for inconsistencies. We take note of some limitations of the algorithm [16]:

- the exponents of the leading-order term needs to be a negative integer or a nonintegral rational number,
- the resonances have to be rational and real numbers,
- excluding the resonance $s = -1$, for a right Painlevé series the resonances must be nonnegative, while for a left Painlevé series, the resonances must be nonpositive.
- for a full Laurent expansion the resonances are mixed [5].

2.1. An illustrative example

To view singularity analysis, consider the Painlevé-Ince equation,

$$y'' + 3yy' + y^3 = 0, \tag{13}$$

see [19, 27].

It is well-known that this equation is maximally symmetric and admits 8 Lie point symmetries [20]. We determine the leading-order of the above equation by firstly substituting $y(x) = a_0(x - x_0)^p$ into (13). From this, we find the values of p that balances the equation, we take the powers of $(x - x_0)$ and equate them to solve for p, i.e $p = -1$.

Consequently, from the same substitution above, we obtain a_0, viz. $a_0 = 1$ or $a_0 = 2$. This implies that the movable singularity is a simple pole and there are two possibilities for the leading-order behaviour. The arbitrary location of the movable singularity gives one of the constants of integrations.

Since (13) is a second-order equation, the second constant of integration has to be determined from a series developed about the singularity.

Now for $a_0 = 1$, we take the truncated Laurent series given by (12) and substitute into (13), to find an expression involving powers of m.

From this expression, we take the coefficients of m and set it to be equal to zero, to get the determining equation of the resonances, viz. $s^2 - 1 = 0$ which solves as $s = 1$ or $s = -1$. The value $s = 1$, gives the term in the series at which the second constant of integration occurs.

For $s = 1$ we use the right Painlevé series (10) which is substituted into (13) and we test for consistency by solving for some a_κ's in the sum.

Hence, taking

$$y(x) = (x - x_0)^{-1} + a_1 + a_2(x - x_0)^1 + a_3(x - x_0)^2,$$

and substituting it into (13), and then separating according to powers of $(x - x_0)$, we have

$$3a_1^2 + 3a_2 = 0 \implies a_2 = -a_1^2,$$

and

$$a_1^3 + 9a_1a_2 + 8a_3 = 0 \implies a_3 = a_1^3.,$$

etc.

Thus, a_1 is an arbitrary constant and (13) passes the Painlevé test with general solution

$$y(x) = \frac{1}{x - x_0} + a_1 - a_1^2(x - x_0) + a_1^3(x - x_0)^2 + \ldots .$$

For $a_0 = 2$, one can do a similar analysis to the above.

3. Third-order boundary flow equations

A selective list of third-order ordinary differential equations that are included in the general equations defined above are discussed. This list is by no means exhaustive and the reader may find many other equations that belong to the given classes. The point is to provide a general framework of analysis for the symmetry and singularity analysis of such equations. Some important equations of (1) are

$$2y''' + yy'' = 0, \tag{14}$$

$$y''' + yy'' - y'^2 = 0, \tag{15}$$

$$y''' + yy'' - y'^2 - M^2 y' = 0, \quad M \text{ is a magnetic parameter,} \tag{16}$$

$$y''' + yy'' - \beta y'^2 = 0, \quad \beta \text{ is a constant,} \tag{17}$$

where, for example, (14) is the Blasius equation, (15)-(16) are Blasius type equations and (17) is the Falkner-Skan equation. Some of the canonical Chazy $I - III$ equations

$$\begin{aligned} I: &\qquad y''' + 6y'^2 = 0, \\ II: &\qquad y''' + 2yy'' + 2y'^2 = 0, \\ III: &\qquad y''' - 2yy'' + 3y'^2 = 0, \end{aligned} \tag{18}$$

also belong to this class.

For some special investigations regarding the Chazy equations, see [1, 28] and references therein.

For $k = 0$ in (1), the reductions associated with Prandtl's boundary layer equations are recovered [22]. Eqs. (2) or (3) may be used to describe, inter alia, the canonical Chazy $IV - XII$ equations

$$\begin{aligned}
IV: &\quad y''' + 3yy'' + 3y'^2 + 3y^2y' = 0,\\
V: &\quad y''' + 2yy'' + 4y'^2 + 2y^2y' = 0,\\
VI: &\quad y''' + yy'' + 5y'^2 + y^2y' = 0,\\
VII: &\quad y''' + yy'' + 2y'^2 - 2y^2y' = 0,\\
VIII: &\quad y''' + 6y^2y' = 0,\\
IX: &\quad y''' - 12y'^2 - 72y^2y' - 54y^4 = 0,\\
XI: &\quad y''' + 2y^2y'' + 2y'^2 - \frac{24}{N^2-1}(y' + y^2)^2 = 0,\ N\text{constant}\\
XII: &\quad y''' - 2y^2y'' + 3y'^2 + \frac{4}{N^2-36}(6y' - y^2)^2 = 0,
\end{aligned} \tag{19}$$

or many related such third-order equations.

An example for the class (4), is the higher-order Lane-Emden equation [29]

$$y''' + \frac{8}{x}y'' + \frac{12}{x^2}y' + y^n = 0. \tag{20}$$

The standard Lane-Emden equation (which is of second-order) $y'' + \frac{k}{y}y' + y^n = 0$, has been shown to pass the singularity test [15].

3.1. Point symmetry classifications

As one will observe through the classification below and the next subsection, symmetry analysis will complement the results obtained through singularity testing, in terms of selecting free parameters of the equations.

We recall the fact that equations with a high degree of symmetry are more useful. The symmetry classification of (1) is:

$$
\begin{aligned}
&\partial_x,\ \text{for } c=0,\ \text{or } c=-\frac{3}{2}b,\ \text{or } h=0,\ \text{or } b,c,h,k\neq 0, c\neq -\frac{3}{2}b.\\
&\partial_x,\ \partial_y,\ \text{for } b=0,\ \text{or } b,h=0.\\
&\partial_x,\ F(y)\partial_y,\ \text{for } b,c,h=0.\\
&\partial_x,\ x\partial_x-y\partial_y,\ \text{for } k=0,\ \text{or } k,c=0,\ \text{or } h,k=0.\\
&F(x)\partial_x, G(y,x)\partial_y,\ \text{for } b,c,h,k=0.\\
&\partial_x,\ \partial_y,\ x\partial_x-y\partial_y,\ \text{for } b,k=0,\ \text{or } b,h,k=0.\\
&\partial_x,\ x^2c\partial_x-(2cxy+18h)\partial_y,\ x\partial_x-y\partial_y,\ \text{for } c=-\frac{3}{2}b,k=0,\\
&\partial_x,\ x^2\partial_x-2xy\partial_y,\ x\partial_x-y\partial_y,\ \text{for } c=-\frac{3}{2}b,k,h=0,\\
&\partial_x,\ \partial_y,\ y\partial_y,\ \cos\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y,\ \sin\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y,\ y\sqrt{k}\cos\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y\\
&+\sqrt{h}\sin\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_x,\ y\sqrt{k}\sin\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y-\sqrt{h}\cos\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_x,\\
&\text{for } b,c=0.
\end{aligned}
\tag{21}
$$

Concerning some of the main Lie bracket relations, we have

$$
\begin{aligned}
[\partial_x, x\partial_x-y\partial_y] &= \partial_x,\\
[\partial_x,\partial_y] &= 0, \quad \text{and}\\
[\partial_y, x\partial_x-y\partial_y] &= -\partial_y.
\end{aligned}
$$

In the specific cases Eqs. (14), (15) and (16) ($M=0$), the Lie point symmetries are $\partial_x,\ x\partial_x-y\partial_y$, while for $M\neq 0$ in (16) we only have ∂_x.

Eq. (17) admits the same two symmetries for $\beta\neq\frac{3}{2}$ but an extra third symmetry

$$(xy-6)\partial_y-\frac{1}{2}x^2\partial_x \ \text{ for } \beta=\frac{3}{2}.$$

The Lie point symmetry classification of (2) is as follows:

$$\partial_x, \quad \text{for} \quad c=0 \quad \text{or} \quad d=\frac{1}{4}b^2+\frac{1}{6}bc \quad \text{or} \quad d=\frac{7}{36}b^2+\frac{1}{3}bc$$
$$\text{or} \quad b=c=0 \quad \text{or} \quad c=f=0,$$
$$\text{or } c\neq 0, d\neq \frac{1}{4}b^2+\frac{1}{6}bc, d\neq \frac{7}{36}b^2+\frac{1}{3}bc, f\neq 0.$$
$$\partial_x, \quad -x\partial_x+y\partial_y \quad \text{for} \quad f=0 \text{ or } \quad d=\frac{7}{36}b^2+\frac{1}{3}bc$$
$$\text{or} \quad c=f=0 \quad \text{or} \quad b=c=f=0.$$
$$\partial_x, \; x\partial_x-2y\partial_y \quad \text{for} \quad b=c=d=0.$$
$$\partial_x, \; \partial_y, \; x\partial_x, \; x\partial_y, \; x^2\partial_y, \; y\partial_y, x^2\partial_x+2xy\partial_y \quad \text{for} \quad b=c=d=f=0.$$

For class (3) , the Lie point symmetries may be classified as:

$$\partial_x, \quad -x\partial_x+y\partial_y, \quad \text{for} \quad b=-\frac{22}{27}nl^2, \quad \text{or} \quad c=\frac{11}{9}nl^2, \quad \text{or} \quad q=\frac{11}{162}l^3n,$$
$$\text{or } b=0, \quad \text{or} \quad b=-3l^2n+3c, \quad \text{or} \quad c=-\frac{1}{2}\frac{-2bl^2n-24lnq+3b^2}{b},$$
$$\text{or} \quad b=0, \; c=nl^2, \quad \text{or} \quad b=l=0, \quad \text{or} \quad b=c=l=0,$$
$$\text{or} \quad b\neq 0, b\neq -3l^2n+3, c\neq -\frac{1}{2}\frac{-2bl^2n-24lnq+3b^2}{b}.$$
$$\partial_x, \; \partial_y, \; x\partial_x-y\partial_y, \quad \text{for} \quad b=n=0, \quad \text{or} \quad b=q=0.$$
$$\partial_x, \quad -(-l^2nx^2+cx^2)\partial_x+(2l^2nxy-2cxy-18)\partial_y, -x\partial_x+y\partial_y,$$
$$\text{for} \quad c=-nl^2-\frac{3}{2}b, q=0.$$
$$\partial_x, \quad -x^2c\partial_x+(-2cxy-18)\partial_y, \quad -x\partial_x+y\partial_y, \quad \text{for} \quad c=-\frac{3}{2}b, n=0.$$
$$\partial_x, \quad -x^2q\partial_x+(-2qxy+l)\partial_y, \quad -x\partial_x+y\partial_y,$$
$$\text{for} \quad c=-\frac{3}{2}b, \; q=-\frac{1}{12}lb \quad \text{or} \quad b=-3l^2+3c.$$
$$\partial_x, \; \partial_y, \; x\partial_x, x\partial_y, \; x^2\partial_y, \; y\partial_y, \; x^2\partial_x+2xy\partial_y, \quad \text{for} \quad b=c=n=0$$
$$\text{or } b=q=0, \; c=nl^2.$$

The individual point symmetry properties of the Chazy equations are well-known to be the translation symmetry ∂_x and the scaling symmetry $y\partial_y-x\partial_x$ that belongs to all equations $I-XII$.

Equations I and II possess ∂_y as a third symmetry, and $(\frac{1}{3}xy+1)\partial_y - \frac{1}{6}x^2\partial_x$ is the extra symmetry for III and XII. The point symmetries of (4) are difficult to obtain for arbitrary functions $b(x)$ and $c(x)$, but for the specific case of the higher-order Lane-Emden equation, we have that if n is arbitrary, the only admitted symmetry is

$$Z = x\partial_x - \frac{3y}{n-1}\partial_y.$$

For the value $n = 1$, we have the four Lie point symmetries

$$\begin{aligned} Z_1 &= y\partial_y, \\ Z_2 &= {}_0\mathrm{F}_2(\ ;\ \frac{5}{3}, 2;\ -\frac{x^3}{27})\partial_y, \\ Z_3 &= \frac{1}{x^2}{}_0\mathrm{F}_2(\ ;\ \frac{1}{3}, \frac{4}{3};\ -\frac{x^3}{27})\partial_y, \\ Z_4 &= \frac{1}{x^3}G^{3,0}_{0,3}\left(\frac{x^3}{27}\,\Big|\,_{1,\frac{1}{3},0}\right)\partial_y, \end{aligned}$$

${}_\mu F_\nu$ and $G^{\mu,\nu}_{\rho,\psi}$ is the generalized hypergeometric and Meijer G function, respectively.

3.2. Singularity analysis

Many of the nonlinear equations which arise in boundary flow investigations and related areas are subjected to numerical investigations, simply because integrability testing is laborious. We tackle this problem with specific reference to the above types of fluid models and explore how free parameters affect integrability.

Since we are dealing with third-order equations, the second and third constants of integration have to be determined from a series developed about the singularity.

A singularity analysis of (1), reveals that $p = -1$ where all terms are dominant excluding the last term in the equation. This p value yields the leading-order behaviour

$$a_0 = 6\,\frac{h}{2\,b + c}.$$

Substitution of the expression

$$y\,(x) = 6\,\frac{h}{(2\,b + c)\,(x - x_0)} + m\,(x - x_0)^{-1+s}\,,$$

into the dominant terms of (1), gives a polynomial in m.

Taking the terms linear in m as equal to zero, we find the equation

$$2\,(s+1)\,\big((b + c/2)\,s^2 + (-4\,b - 7/2\,c)\,s + 6\,b + 3\,c\big)\,h = 0,$$

which one solves to find the three resonances

$$s = -1,\;\; s = \frac{1}{4\,b + 2\,c}\left(8\,b + 7\,c + \pm\sqrt{-32\,b^2 + 16\,bc + 25\,c^2}\right).$$

The acceptable values for the resonance s, must be real and rational.

Note that there are many possibilities for the second and third resonance to be positive, negative or complex.

To establish the particular values of the resonance and further progress in the method, we may consider particular values of the free parameters, which also facilitates the consistency test for the constants of integration.

Testing equation (14) - (16) leads to complex conjugate resonances, so we conclude that the Painlevé test is unreliable in these cases. For equation (17), we find that the second and third resonances are

$$s = \frac{1}{4 - 2\,\beta}\left(8 - 7\,\beta + \pm\sqrt{25\,\beta^2 - 16\,\beta - 32}\right).$$

Taking $\beta = -1$, we have that $s = 2$ and $s = 3$. Regarding the consistency test, we substitute the truncated right Painlevé series

$$y\,(x) = \frac{2}{x - x_0} + a_1 + a_2(x - x_0) + a_3(x - x_0)^2,$$

into (17), and find that $a_1 = 0$ while a_2 and a_3 are arbitrary constants.

Hence, we have the correct number of constants and we conclude that that Falkner-Skan equation passes the singularity test.

As for (2), all terms are dominant except the last term, and one finds $p = -1$ with the leading-order behaviour:

$$na_0 = 0, \frac{1}{2\,d}\left(2\,b + c \pm \sqrt{4\,b^2 + 4\,bc + c^2 - 24\,d}\right), \quad d \neq 0. \tag{22}$$

Taking the positive square root in a_0, we obtain the resonances

$$s = -1, s = -\frac{1}{4\,d}\left(b\sqrt{z} + 2\,b^2 + bc \pm \sqrt{w} - 14\,d\right),$$

where

$$\begin{aligned} w \;=\;& 4\sqrt{z}b^3 + 2\sqrt{z}b^2c + 8\,b^4 + 8\,b^3c + 2\,b^2c^2 - 12\sqrt{z}bd + 8\sqrt{z}cd \\ & -48\,b^2d + 4\,bcd + 8\,c^2d + 4\,d^2, \end{aligned}$$

and

$$z = 4\,b^2 + 4\,bc + c^2 - 24\,d.$$

Alternatively, the negative square root in a_0, provides the resonances

$$s = -1, s = \frac{1}{4\,d}\left(\sqrt{z}b - 2\,b^2 - bc + 14\,d \pm \sqrt{\omega}\right),$$

where

$$\begin{aligned} \omega \;=\;& -4\sqrt{z}b^3 - 2\sqrt{z}b^2c + 8\,b^4 + 8\,b^3c + 2\,b^2c^2 + 12\sqrt{z}bd - 8\sqrt{z}cd \\ & -48\,b^2d + 4\,bcd + 8\,c^2d + 4\,d^2, \end{aligned}$$

Let us take VI as an example. Applying the formula (22), one finds $a_0 = 0, 1, 6$.

Suppose we choose $a_0 = 1$, then for the resonances we have $s = -1, 1, 5$. Testing for consistency, one finds that VI passes the Painlevé test.

For Eq. (3), all terms are dominant with $p=-1$, and a_0 has four solutions, namely 0 (which we ignore), a pair of complex conjugates ($nq \neq 0$)

$$a_0 = -\frac{2\,l}{3\,q} \pm \frac{i}{2}\sqrt{3}\left(\frac{1}{3\,nq}\sqrt[3]{t} - \frac{l^2n+6\,b+3\,c}{3\,q}\frac{1}{\sqrt[3]{t}}\right) - \frac{1}{6\,nq}\sqrt[3]{t} - \frac{l^2n+6\,b+3\,c}{6\,q}\frac{1}{\sqrt[3]{t}}, \tag{23}$$

and

$$a_0 = \frac{1}{3\,nq}\sqrt[3]{t} + \frac{l^2n+6\,b+3\,c}{3\,q}\frac{1}{\sqrt[3]{t}} - \frac{2\,l}{3\,q}, \tag{24}$$

where we have defined

$$t = \left(l^3n - 18\,bl - 9\,cl + 3\,v - 81\,q\right)n^2,$$

and

$$\begin{aligned} v^2 \quad = \quad -\frac{3}{n}\Bigg(& 2\,bl^4n^2 + cl^4n^2 + 6\,l^3n^2q - 8\,b^2l^2n - 8\,bcl^2n - 2\,c^2l^2n \\ & -108\,blnq - 54\,clnq + 8\,b^3 + 12\,b^2c + 6\,bc^2 + c^3 - 243\,nq^2\Bigg). \end{aligned}$$

Thus the three formulae for a_0 form three cases for the leading-order. Utilizing the value of a_0 in Eq. (24), we find for the resonances $s_0=-1$, and the formula for s_1, viz.

$$\begin{aligned} \left(\sqrt[3]{t}\left(-6\,l^3n^2q + 6\,q\,(18\,bl + 9\,cl - 3\,v + 81\,q)\,n\right)\right)\,s_1 = \\ -18\,b\left(-1/18\,l^3n + bl + 1/2\,cl - v/6 + 9/2\,q\right)t^{2/3} + \\ 36\,\left(-1/18\,l^3n + bl + 1/2\,cl - v/6 + 9/2\,q\right) \\ (bl + 21/2\,q)\,n\sqrt[3]{t} - 486\,\left(1/6\,nl^2 + b + c/2\right)bnq - 108\,nb^3l \\ -108\,\left(1/9\,l^3n + cl - v/6\right)nb^2 \\ -27\,\left(1/3\,nl^2 + c\right)\left(-1/9\,l^3n + cl - v/3\right)nb \\ +\Bigg(\frac{1}{n}\Big(-2592\,n\Big(-\frac{l^6n^3}{216} + \frac{5\,l^3n^2}{12}\left((b + c/2)\,l - v/15 + \frac{18\,q}{5}\right) \end{aligned}$$

$$+\Big(-5/2\ (b+c/2)^2\, l^2 + 1/2\ (b+c/2)\,(v-54\,q)\, l + 9/4\, q\,(v-27\,q)\Big)\, n$$
$$+ (b+c/2)^3\,)$$
$$\left(\left(1/2\, l^2 b^2 - 9\left(b+\frac{8\,c}{9}\right) ql - \frac{69\,q^2}{4}\right) n + b^2\,(b+c/2)\right) t^{2/3}$$
$$-216\, b^2\big(-\frac{l^6 n^3}{216} + \frac{5\,l^3 n^2}{12}\left((b+c/2)\, l - v/15 + \frac{18\,q}{5}\right) +$$
$$\big(-\frac{5}{2}\ (b+c/2)^2\, l^2 + 1/2\ (b+c/2)$$
$$(v-54\,q)\, l + 9/4\, q\,(v-27\,q)\,\big) n + (b+c/2)^3\,)$$
$$t^{4/3} - 7776\, n^2\Big(\big(\frac{l^7 n^4}{1944}\left(l^2 b^2 - \frac{531\,ql}{2}\left(b+\frac{92\,c}{177}\right) + 6\,q\,(v-135\,q)\right)$$
$$-\frac{7\,l^4 n^3}{162}\big(b^2\,(b+c/2)\, l^3 + \left(\left(-v/14 - \frac{1479\,q}{14}\right) b^2 - \frac{459\,qcb}{4} - 30\,c^2 q\right) l^2 +$$
$$\frac{279\,ql}{28}\left(\left(v-\frac{2646\,q}{31}\right) b + \frac{50\,c}{93}\left(v-\frac{2079\,q}{25}\right)\right) + \frac{243\,q^2\,(v-45\,q)}{7}\big)$$
$$-1/27\, l\Big(\big((v+534\,q)\, b^2 + 588\,qcb + 147\,c^2 q\big)\,(b+c/2)\, l^3$$
$$-\frac{447\,ql^2}{4}\big(\left(v-\frac{11853\,q}{149}\right) b^2$$
$$+\frac{165\,bc}{149}\left(v-\frac{378\,q}{5}\right) + \frac{43\,c^2}{149}\left(v-\frac{3213\,q}{43}\right)\big) - \frac{8505\,q^2 l}{8}\big(\left(v-\frac{1377\,q}{35}\right) b$$
$$+\frac{8\,c}{15}\left(v-\frac{270\,q}{7}\right)\big) - 2187\,q^3\,(v-27\,q)\,\Big)$$
$$n^2 + \frac{(14\,b+7\,c)\,n}{9}\big(b^2\,(b+c/2)^2\, l^3 - 3/14\ (b+c/2)\,\big((v-87\,q)\, b^2$$
$$-\frac{17}{2}\,qcb + 4\,c^2 q\big) l^2$$
$$-\frac{15\,ql}{14}\left(\left(v-\frac{81\,q}{2}\right) b^2 - \frac{19\,bc}{20}\left(v-\frac{810\,q}{19}\right) - 1/2\left(v-\frac{243\,q}{5}\right) c^2\right)$$
$$+\frac{81\,q^2\,(v-27\,q)\,(b+4\,c)}{56}\big) - 2/3\ (b+c/2)^4\,\big(b^2 l - 3/2\,q\,(b+4\,c)\big)\,\big)\sqrt[3]{t}$$
$$+\frac{l^8 n^5}{2592}\left(l^2 b^2 + 366\left(b+\frac{88\,c}{183}\right) ql - 8\,q\,(v-135\,q)\right)$$
$$-\frac{131\,l^5 n^4}{1296}\big(b^2\,(b+c/2)\, l^3$$

$$
\begin{aligned}
&+\left(\left(-\frac{5\,v}{131}+\frac{6936\,q}{131}\right)b^2+\frac{5748\,qcb}{131}+\frac{1248\,c^2q}{131}\right)l^2\\
&-\frac{591\,ql}{131}\left(\left(v-\frac{16389\,q}{197}\right)b+\frac{88\,c}{197}\left(v-\frac{918\,q}{11}\right)\right)-\frac{1944\,q^2\,(v-45\,q)}{131}\bigg)\\
&+\frac{259\,l^2n^3}{108}\left(b^2\,(b+c/2)^2\,l^4\right.\\
&-\frac{\left(53\,b+\frac{53\,c}{2}\right)l^3}{518}\left((v-156\,q)\,b^2-\frac{1302\,qcb}{53}+\frac{336\,c^2q}{53}\right)-\frac{2307\,ql^2}{1036}\\
&\left(\left(v-\frac{52623\,q}{769}\right)b^2\right.\\
&+\frac{445\,bc}{769}\left(v-\frac{6345\,q}{89}\right)+\frac{64\,c^2}{769}\left(v-\frac{513\,q}{8}\right)\bigg)-\frac{4374\,q^2l}{259}\left(\left(v-\frac{77\,q}{2}\right)\right.\\
&b+\frac{10\,c}{27}\left(v-\frac{189\,q}{5}\right)\bigg)-\frac{8748\,q^3\left(v-27\,q\right)}{259}\bigg)\\
&+\left(-\frac{175\,b^2\,(b+c/2)^3\,l^4}{18}+\frac{71\,\,(b+c/2)^2\,l^3}{36}\left(\left(v-\frac{4920\,q}{71}\right)b^2\right.\right.\\
&+\frac{2460\,qcb}{71}+\frac{1056\,c^2q}{71})+\frac{(38\,b+19\,c)\,ql^2}{3}\\
&\left(\left(v-\frac{2187\,q}{76}\right)b^2-\frac{377\,bc}{304}\left(v-\frac{27945\,q}{377}\right)-1/2\left(v-\frac{1458\,q}{19}\right)c^2\right)\\
&-9/2\left((v-108\,q)b^2+31\left(v-\frac{2403\,q}{62}\right)cb+13\left(v-\frac{513\,q}{13}\right)c^2\right)q^2l\\
&-\frac{243\,q^3\left(v-27\,q\right)\left(b+4\,c\right)}{4}\bigg)n^2+\frac{23\,\,(b+c/2)^2\,n}{6}\\
&\left(b^2\,(b+c/2)^2\,l^2+(v-54\,q)\,l\right.\\
&+9/4\,q\,(v-27\,q)\,\big)n+(b+c/2)^3\,\big)1/23\,\left((v-102\,q)\,b^2-291\,qcb-120\,c^2q\right)\\
&(b+c/2)+\frac{33\,q}{46}\left(\left(v-\frac{405\,q}{11}\right)b^2+\frac{9\,c\,(v-81\,q)\,b}{11}+4/11\,c^2\,(v-81\,q)\right)\bigg)\\
&\left.+b^2\,(b+c/2)^5\,\big)\big)\right)^{1/2}.
\end{aligned}
$$

The third resonance can be found from the following relation involving s_1

$$
s_2=-2\,\frac{1/6\,bt^{2/3}+n\left((-1/3\,bl-7/2\,q)\,\sqrt[3]{t}+b\left(1/6\,nl^2+b+c/2\right)\right)}{\sqrt[3]{t}qn}-s_1.
$$

It is obvious that a general formulae for the resonances of the other two leading-order's, Eq. (23), will also be enormous, and so we omit displaying them here. However they can be recovered from substituting

$$y(x)=\frac{a_0}{x-x_0}+m\,(x-x_0)^{-1+s}\,,$$

where a_0 is given by (23), into Eq. (3) and extracting the coefficients of m, which must be equated to zero.

As examples, suppose we apply the above formulae to the Chazy equations. Due to the restriction $nq \neq 0$, the above formulae will only apply to certain cases of the Chazy equations.

In all other cases of the Chazy equations, the singularity formulae related to Eqs. (1) and (2) apply.

As for XI, we find the cases:

- Case i: $a_0 = 1$, with resonances $s = -1, 2, 3$,
- Case ii: $a_0 = \frac{N}{2} + \frac{1}{2}$, with resonances $s = -1, 6, N$,
- Case iii: $a_0 = -\frac{N}{2} + \frac{1}{2}$ with resonances $s = -1, 6, -N$.

For Case i, we find that the equation passes the singularity test ($N \neq 1$) with the right Painlevé series

$$y(x)=\frac{1}{x-x_0}+a_2(x-x_0)+a_3(x-x_0)^2+\ldots.$$

N must be an integer for us to have integer resonances in Cases ii and iii, which is a strong indication that the equation does not possess the Painlevé property otherwise. If $N = 0$, the equation fails the consistency test.

Hence we conclude, that the equation passes the Painlevé test, for $1 < N \in \mathbb{N}$, provided that $N \neq 1$.

In the case of XII, we recover the results of [9], whereby the equation passes the Painlevé test.

Finally, we get to equations of the form (4). In this case, only the first and last term are dominant, so that $p = -3\,(n-1)^{-1}$ for $n > 1$ and the leading-order is

$$a_0 = \left(3\,\frac{(n+2)\,(2\,n+1)}{k\,(n-1)^3}\right)^{(n-1)^{-1}}. \tag{25}$$

As before, in solving for the resonances s, one must extract the coefficients of m in and set them to zero, This separation of m may only be possible if one knows the value of n, so this is as far as we may go with the calculation.

To proceed with the analysis, we may consider specific examples.

Suppose we take the higher-order Lane-Emden equation (20). Solving, will give the following resonances

$$s = \frac{3}{n-1}, \quad s = \frac{n+2}{n-1}, \quad s = \frac{2n+1}{n-1}.$$

Since we know that one of the s variables should be equal to -1, then we can find all the possible values of n.

Hence, we get $n = 0,\ -1/2,\ -2$. Clearly all these values of n give p as a positive integer, where we would prefer that the value of p should be negative.

Suppose we ensure that p is a negative integer, then we may select $n = 2$, and we will have $p = -3$. For this case, with formula (25) we find $a_0 = 60$, and we must solve

$$s^2 - 13s + 60 = 0, \tag{26}$$

which unfortunately gives complex conjugate resonances. Another possible choice is the value $n = 4$, which yields $p = -1$. Here, using formula (25), we have that $a_0 = 6^{\frac{1}{3}}$ and we find the equation

$$s^2 - 7s + 18 = 0, \tag{27}$$

yielding complex conjugate resonances. In such situations the reliability of the Painlevé test is uncertain and the analysis is inconclusive.

4. Conclusion

In this paper we have sought to simplify the integrability testing of several classes of fluid equations, using the notion that there is strong evidence that integrable equations possess the Painlevé property. In essence of the method, one looks at the existence of a Laurent series for each dependent variable of the equation.

We reviewed three general classes of third-order equations with constant coefficients, and extended the study to a nonconstant coefficient class. In particular, we prescribed an outline for the singularity testing of these classes of equations.

Integrable equations and systems are a rarity in practice. Many scientists are dissuaded from testing for integrablility, in favour of resorting to numerical methods. This need not be the case, especially in situations where a given equation fits into our analysis, owing to the formulae described above.

Moreover, we aimed to promote the use of Lie and Painlevé analyses as a basis for selecting equation parameters. This was observed from the effects of the parameters on the symmetry properties of the equations and particularly in the structure of the singularity analysis. That is, the results obtained above illustrate the constraint effects of the parameters of the equation, on integrability requirements as per Painlevé tests.

Another direct consequence of this idea, is that even in the case of equations that fail the singularity test, singularity testing can yield a great deal of information regarding the equation's free parameters. Consequently, the dual analyses of Lie and Painlevé offers the prospect of providing better insight about the equation under consideration.

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School of Mathematics
University of the Witwatersrand
Johannesburg Wits 2001
South Africa
E-mail: Sameerah.Jamal@wits.ac.za

School of Mathematics
University of the Witwatersrand
Johannesburg Wits 2001
South Africa
E-mail: jolletkubayi@gmail.com

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